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HOW DOES A FRACTION GET ITS NAME?[1]

COMO UMA FRAÇÃO RECEBE SEU NOME?

Arthur B. Powell[2]

Abstract: Philosophical and cultural perspectives shape how a fraction is named and defined. In turn, these perspectives have consequences for learners' conceptualization of fractions. We examine historical foundations of two perspectives of what are fractions—partitioning and measuring—and how these views influence fraction knowledge. For the dominant perspective, partitioning, we indicate how its approach to what is a fraction that discretizes objects and its well-meaning visual correlates cause learners a host of perceptual difficulties. Based on the human cultural and social practice of measuring continuous quantities, we then offer an alternative understanding of what is a fraction and illustrate the promise of this view for fraction knowledge. We introduce pedagogical tools, Cuisenaire rods, and illustrate how they can be used to implement a measuring perspective to comprehending properties and a definition of fractions. We end by sketching how to initiate a measuring perspective in a mathematics classroom.

Keywords: Fractions; Gattegno; Measuring; Partitioning; Unit fractions.

Resumo: Perspectivas filosóficas e culturais moldam como uma fração é nomeada e definida. Por sua vez, essas perspectivas têm consequências para a conceitualização de frações dos estudantes. Examinamos os fundamentos históricos de duas perspectivas do que são frações—particionamento e medição—e como essas visões influenciam o conhecimento das frações. Para a perspectiva dominante, partição, indicamos como sua abordagem ao que é uma fração que discretiza objetos e seu correlato visual bem-intencionado causa aos alunos uma série de dificuldades perceptivas. Com base na prática cultural e social humana de medir quantidades contínuas, oferecemos um entendimento alternativo do que é uma fração e ilustramos a promessa dessa visão para o conhecimento da fração. Introduzimos ferramentas pedagógicas, varas Cuisenaire e ilustramos como elas podem ser usadas para implementar uma perspectiva de medição para compreender propriedades e uma definição de frações. Terminamos esboçando como iniciar uma perspectiva de medição em uma sala de aula de matemática.

Palavras-chave: Frações; Gattegno; Medição; Partição; Frações unitárias.

1 Introduction

A response to the question—how does a fraction get its name? — may seem unproblematic and straightforward. A schooled individual might reply, "a fraction is parts of a whole." Indeed, this is the viewpoint found in textbooks approved for schools (SCHEFFER; POWELL, 2019) and appears in other authoritative sources such as

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[2] Professor Titular, Rutgers University. Department of Urban Education, Rutgers University-Newark, Newark, New Jersey, USA. E-mail: powellab@newark.rutgers.edu

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physical dictionaries and Internet sites. Nevertheless, as simple as a fraction may appear, ideas about *what it is* and *how to define it* divide into two distinct perspectives: partitioning and measuring. The first perspective emphasizes counting discrete objects and the second comparing continuous quantities. Each view has its historical origin and epistemological consequence. What follows is a brief description of the birth of each perspective and how each influences fraction knowledge. Afterward, as the partitioning perspective is well known, the measuring perspective's influence on fraction understanding is more extensively described.

2 Partitioning Perspective

The current dominant understanding of how a fraction gets its name is rooted in a relatively recent development in the history of mathematics. This historical development occurred at the beginning of the 20th-century and is based on a philosophical view championed by the influential German mathematician, David Hilbert, called formalism. Formalists believed that all mathematics can be formulated based on rules for manipulating formulas without any reference to the meanings of the formulas or practical contexts. That is, formalists contend that the primary objects of mathematical thought are the mathematical symbols themselves and not any meanings ascribed to them (SIMONS, 2009). This philosophical belief about the nature of mathematics permeates mathematics education. One consequence of formalism is how rational numbers, especially fractions, are defined (SCHMITTAU, 2003). A formalist definition of rational numbers is the following: Rational numbers represented as common fractions are bipartite symbols that express quotients or ratios of two integers, a/b, such that a and b are integers and $b \neq 0$. In the expression, a/b, a is called the dividend or numerator and b the divisor or denominator.

That is a formal definition of a fraction. However, as such a definition would make little sense to children, mathematics educators devised visual correlates for fractions that involve partitioning everyday items (DAVYDOV; TSVETKOVICH, 1991; SCHMITTAU, 2003) such as pizzas, chocolate candy bars, and chestnuts. After equipartitioning a pizza or chocolate bar or identifying a subset of a collection of chestnuts, a faction's denominator represents a count of the equipartitioned parts or the

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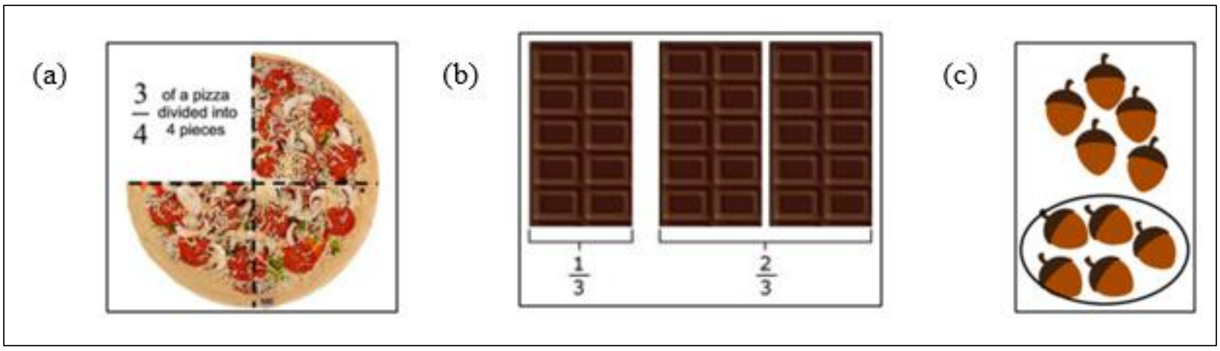
collection, and separately the numerator is a count of the parts of interest or the identified subset (Figure 1).

Figure 1: Three partitioning representations of fractions: (a) A pizza that was divided into four parts, now with the three parts or $3/4$ of it shown. (b) A multi-sectioned chocolate bar equipartitioned into three parts with $1/3$ and $2/3$ of it indicated. (c) A collection of 10 chestnuts with a subset of five chestnuts identified to show $5/10$



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These visual representations supply the formal definition of a fraction with quotidian interpretations. Though such meanings contradict the formalist project, they nevertheless do provide children with visual access to the formalist definition of a fraction and its bipartite symbol, a/b. The meaning of the visual depiction involves dividing an area into discrete equal pieces or identifying a subset of a collection of objects and then a two-fold counting and recording process: (1) the number of equal parts or objects and (2) the number of parts of interest or objects in an identified subset. This view of how a fraction gets its name can be called a partitioning perspective and relies on counting.

For students, this perspective entails cognitive difficulties. One conceptual challenge is that the equipartitioning an object does not bestow meaning to an improper fraction, a fraction whose numerator is larger than its denominator. Mack (1993) documents that students sense improper fractions such as $4/3$ to be meaningless since one cannot have 4 parts of an object that is divided into 3 parts. Another cognitive issue arises from an instructional emphasis on the two-part structure of a fraction's symbolic form. Rather than communicating that a fraction holistically is a single magnitude, the instructional focus suggests to students that a fraction is composed of two distinct numerical parts and primes them to apply inappropriately whole number properties to evaluate fractions. Students confound the following:

1. the number of pieces in a partition with the size of each piece, so $1/4$ is larger than $1/3$ since 4 is larger than 3;

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2. the addition of fractions with adding whole numbers: $1/2 + 1/3 = 2/5$; and

3. the requirement for equal parts, "counting noncongruent parts to name a fraction *one third* in a circle that is partitioned into a half and two fourths" (NI; ZHOU, 2005, p. 29, original emphasis).

These three conceptual mistakes, as well as students' documented reluctance to view improper fractions as meaningful, result from how the partitioning perspective defines a fraction, parts of a single equipartitioned whole.

3 Measuring Perspective

There is another view of how a fraction derives its name. Rather than philosophical, this perspective is cultural. This alternative view of how fractions emerged is based on an understanding of the human social practice of comparing or measuring continuous quantities. More than four millennia ago, in Mesopotamian and Egyptian cultures, along the Tigris, Euphrates and Nile rivers, with the birth of agriculture, the material conditions introduced the need to measure quantities of land, crops, seeds, and so forth and to record the measures (CLAWSON, 1994/2003; STRUIK, 1948/1967). To measure the distances of land, ancient Egyptian surveyors stretched ropes in which the length between two knots represented a unit of measure. From this social practice arose simultaneously geometry and fractional numbers (ALEKSANDROV, 1963; CARAÇA, 1951; ROQUE, 2012). That is, fractions emerged as individuals wanted to know, for instance, the extent of a distance d, in comparison to a unit of measure u. There are two cases. Either d equals an exact multiple u, or it does not, which occasions a need for fractional numbers. In what follows, we introduce a pedagogical tool to help us examine each of the two cases from a measuring perspective.

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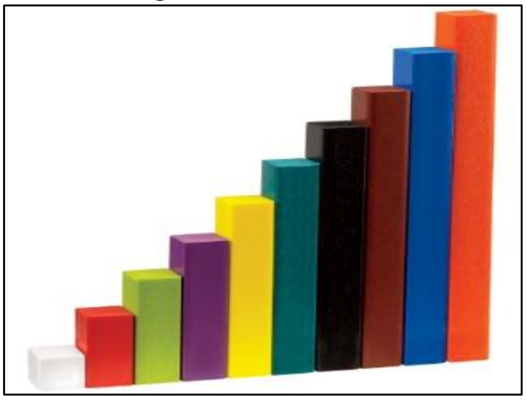
Figure 2: Cuisenaire rods



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In the first case, d equals exactly k units of measure u, where k is a whole number, then $d = k \times u$. To illustrate this expression, consider the Cuisenaire rods (Figure 2). They are measurable quantities that come in ten different colors and sizes. The colors are white, red, green, purple, yellow, dark green, ebony, tan, blue, and orange. In terms of sizes, the length of each different color rod in sequence increases by one centimeter, starting with a cube whose length is one centimeter, where rods of the same length have the same color and *vice versa*. These materials can be used to instantiate a measuring perspective for fraction knowledge, as a particular relation of quantities (GATTEGNO, 1974/2010).[3] On this point, Gattegno (1974/2010), with reference to Cuisenaire rods, summarizes the role of measurement in elementary mathematics:

> Measure, in the work with the rods, is borrowed from physics and introduces counting by the back door, since it is necessary to know *how many* times the unit has been used to associate a number with a given length. But measure is also the source of fractions and mixed numbers, and serves later to introduce real numbers. Thus measure is a more powerful tool than counting, which it uses as a generator of mathematics. Counting … can be interpreted again as being a measure with white rods. Measure is naturally also an interpretation of iteration …. (p. 196, original emphasis).

The Cuisenaire rods have many attributes. One attribute is color, and another is length. Implicit in Gattegno's statement is that length is the attribute of interest and to measure. Consider a tan rod and select as the unit of measure the red rod. What is the tan rod's length in units of red rods? Figure 3a shows that, reading from left to right, the length of one of the tan rods equals the length of four of the red rods. This statement reveals a comparative relation between the two quantities, tan and red rods. The relation

[3] Other researchers have developed and investigated instructional approaches based on the measurement of continuous quantities. See, for example, the work of Brousseau, Brousseau, and Warfield (2004); Carraher (1996); Davydov and Tsvetkovich (1991); Dougherty and Venenciano (2007); Powell (2019); Venenciano and Heck (2016); and Venenciano, Slovin, and Zenigami (2015).

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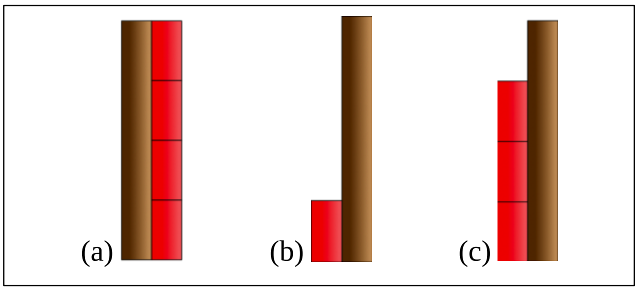
is multiplicative as four of the red rods equals one of the tan rods. This particular multiplicative comparison between the two quantities can be stated differently. Among the possibilities, here are three verbal expressions equivalent to the original statement:

1. One tan rod measures four red rods.
2. When measured by red rods, the tan rod is equivalent to four red rods.
3. One tan rod measured by red rods equals four.

Figure 3: (a) The tan rod equals four red rods. (b) The red rod is one-fourth of the tan rod. (c) Three red rods are three-fourths of the tan rod



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Those are alternative ways to talk about the length of a tan rod when the red rod is the unit of measure. Now, if the length of a tan rod is the unit of measure, what is the red rod's length in units of tan rods? As the length of four red rods equals the length of a tan rod, then in Figure 3b, reading from left to right, the length of one red rod is one-fourth the length of a tan rod. The length of three red rods is then three-fourths the length of a tan rod (Figure 3c). Continuing with the pattern, the length of five red rods is five-fourths the length of a tan rod. And so on. In sum, by comparing the length of two quantities—with one quantity considered as the unit of measure and used to measure the other quantity—is how a fraction gets its name. This understanding can be called a measuring perspective.

The verbal statements associated with the rod configurations in Figure 3 have ways to be represented symbolically, using mathematical notation. For this purpose, each Cuisenaire rod can be expressed with the initial letter of its color name. Table 1 displays correspondences between the color of each rod and a letter to symbolize it.

Table 1: From smallest (1 centimeter) to largest (10 centimeters), the Cuisenaire rods' color-letter correspondence

white	red	green	purple	yellow	dark green	ebony	tan	blue	orange
w	r	g	p	y	d	e	t	b	o

Source: Author

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Below in Table 2, in the left column are listed the verbal statements associated with the rod configurations in Figure 3, and in the right column are the corresponding mathematical symbolic expressions. Some mathematical expressions correspond to more than one verbal statement.

Table 2: Correspondence between verbal and symbolic expressions

Expressions	
Verbal	Symbolic
1. The length of one of the tan rods equals the length of four of the red rods. 2. When measured by red rods, the tan rod is equivalent to four red rods. 3. One tan rod measures four red rods.	$t = 4r$
4. A tan rod measured by red rods equals four.	$\frac{t}{r} = 4$
5. The length of one red rod equals one-fourth the length of a tan rod. 6. Red is one-fourth of tan.	$r = \frac{1}{4} \times t$
7. A red rod measured by tan rods equals one-fourth.	$\frac{r}{t} = \frac{1}{4}$

Source: Author's arquive

Now, the case that precipitates the invention of fractions. When the ancient Egyptian surveyors wanted to know the extent of a distance d, in comparison to a unit of measure u, it was not always the case that d was exactly k units of measure u, where k is a whole number. That is, it is not guaranteed that d, measured by u equals exactly $k \times u$. For example, using Cuisenaire rods, consider the length of the purple rod to be the unit of measure. What is the measure of the length of the orange rod? In Figure 4a, an orange rod does not exactly measure a whole number of times the length of purple rods. It equals two purple and less than another purple rod. Do the length of a purple rod and the length of an orange rod measure a whole number of times the length of another rod? As shown in Figure 4b, a purple rod equals two red rods. One red rod, a portion of the purple rod, is exactly the length that completes the measure of the orange rod (Figure 4c). From Figure 4c it can be determined that an orange rod equals five red rods. Therefore, the red rod is a common subunit of both the orange and purple rods. Since the purple rod is the unit and equals the length of two of the red rods, the length of a red rod is one-half of the length of a purple rod. Also, the red can be called a subunit of the purple rod. In conclusion, when measured by a purple rod and its subunit, the length of an orange rod measures the length of two and one-half of the length of a purple rod. The conclusion in mathematical

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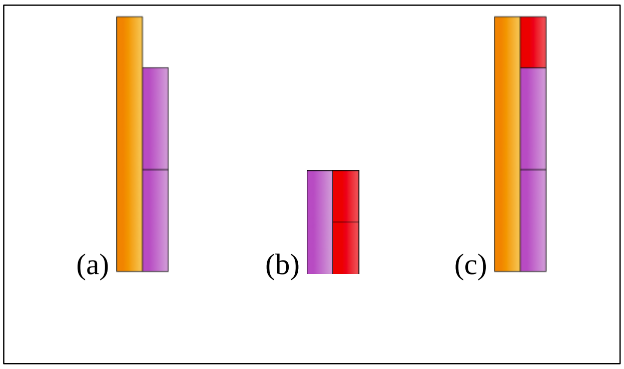
notation is this: $o = 2 \times p + \frac{1}{2} \times p$, $o = 2 \times p + \frac{1}{2} \times p$, or, since each purple rod equals two red rods, this: $o = \frac{5}{2} \times p$.

Figure 4: Measuring the length of the orange rod with the purple rod as the unit of measure



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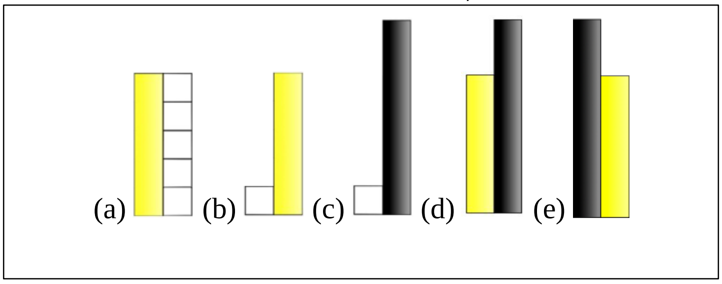
Using Cuisenaire rods, the factional name of a rod's length can be derived using any rod as the unit of measure. For instance, determine the name of the length of an ebony rod measured by a yellow rod. Use the yellow rod to measure the length of a white rod (Figures 5a and 5b). Here, the white rod is used as a subunit of the yellow rod, and its length equals one-fifth of a yellow rod's length, which is written as $w = \frac{1}{5} \times y$. Since seven white rods measure the length of an ebony rod, the white rod is also a subunit of the ebony rod (Figure 5c). A yellow rod is five-sevenths of an ebony rod (Figure 5d), and inversely, its length is seven-fifths of a yellow rod (Figure 5e). Respectively, these verbal expressions are symbolized as follows: $y = \frac{5}{7} \times e$ and $e = \frac{7}{5} \times y$.

Figure 5: (a) $y = 5w$. (b) $w = \frac{1}{5} \times y$, (c) $y = \frac{5}{7} \times e$, and (d) $e = \frac{7}{5} \times y$



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In general, if d does not equal an exact multiple of u, then there may exist a common subunit of measure, v, of both d and u. If this is the case, then d equals exactly m subunits of v, and u equals exactly n subunits of v. Since $u = n \times v$, the length of v

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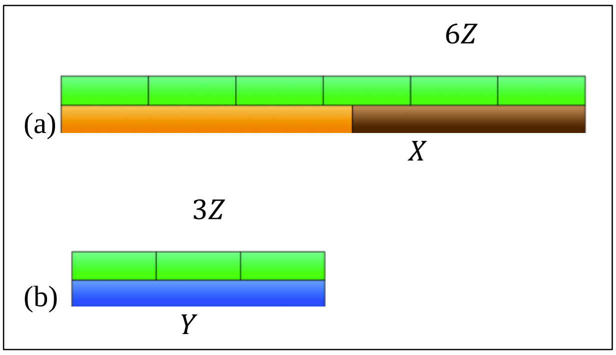
equals one-n^{th} the length of u: $v = \frac{1}{n} \times u$. Also, since $d = m \times v$, the length of v also equals one-m^{th} the length of d: $v = \frac{1}{m} \times d$. Thus, as d equals exactly m subunits of v, it also equals m of $\frac{1}{n} \times u$ or $m \times \frac{1}{n} \times u = \frac{m}{n} \times u$, which implies $d = \frac{m}{n} \times u$. This expression represents a multiplicative comparison between the two quantities d and u.

The fact that there is a multiplicative comparison between the lengths d and u means that the two lengths are commensurable. In mathematics, the commensurability of two different quantities such as lengths, X and Y, indicates that they have a common unit of measure. Possessing a standard unit means that there is a third length, Z, less than or equal to the smaller of X and Y, such that when placed end-to-end a whole number of times creates a length equal to X (Figure 6a). Similarly, when Z is placed end-to-end a different whole number of times, it creates a length equal to Y (Figure 6b). The ratio of X and Y—X/Y—is a fraction.

Figure 6: Lengths X and Y have Z as a common unit of measure, and, therefore, are commensurate



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Sometime after the Egyptians invented fractions and a notation for them,[4] the ancient Greeks discovered that such ratios of lengths were not always commensurable (STRUIK, 1948/1967), meaning measurable by the same unit, leading to the discovery of irrational numbers.[5]

Based on this measuring perspective, a fraction is defined as *a multiplicative comparison between two commensurable quantities of the same kind*.

[4] See Ifrah (1981/1998) or Roque (2012) for information about how ancient Egyptians symbolized fractions.
[5] For discussions about irrational numbers and reflections concerning the mathematics classroom, see Broetto and Santos-Wagner (2017).

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4 Concluding considerations

This definition is derived from a historical understanding of the emergence of fractional numbers and the mathematical implications of measuring lengths for an understanding of rational and irrational numbers. In the ancient past, the material conditions evolved and necessitated language to describe magnitudes whose measures were greater than a whole number but less than its successor. With fractions, these magnitudes of length could more precisely be quantified. Commenting on its cognitive usefulness for mathematics learning, Carraher (1993) observes that "[l]ength, more so than other quantities, expresses magnitude directly and unambiguously. A student can straightforwardly compare two lengths through visual inspection" (p. 284).

Mathematics education researchers such as Carraher (1993) and Gattegno (1974/2010) have observed that the comparison of length, a continuous quantity, is conceptually straightforward. This view corresponds to recent results in cognitive neuroscience. There is increasing evidence that humans have an innate capacity to discern the relative magnitude among ratios of nonsymbolic continuous quantities (MATTHEWS; ZIOLS, 2019). Functional neuroimaging studies have consistently identified overlap brain regions involved in comparing nonsymbolic ratios and symbolic fractions (JACOB; NIEDER, 2009; MOCK *et al.*, 2019; MOCK *et al.*, 2018). As Matthews and Ellis (2018) and Matthews and Ziols (2019) underscore, this nonsymbolic capacity to discriminate magnitudes of continuous ratios remains to be recruited and investigated in the context of fraction instruction. Such an investigation is facilitated with a measuring perspective to fraction knowledge (POWELL, 2019).

The measuring perspective has several theorized mathematical and cognitive outcomes worth underscoring. It can instigate in students the following mathematical and cognitive awarenesses:

- When the quantity to be measured is not a whole-number multiple of the unit of measure, the need for a subunit of measure that is commensurable with both the unit of measure and the quantity to be measured;
- Lengths can be expressed as improper fractions or mixed numbers;
- The magnitude of a quantity can be expressed by different fractional numbers as a consequence of different units of measure;

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- For a given length, different units of measure yield different measures—the smaller the magnitude of the unit, the larger the measure; and
- Contrary to the partitioning perspective—where the act of counting parts of a single quantity suggests an additive relation—a fraction is a multiplicative relation between two commensurable quantities; the relation is comparison.

The first two of these awarenesses provide insights into a definition of a fraction from a measuring perspective. The definition can be formulated in two parts, defining first a unit fraction and then a general fractional number:

For a given unit of measure and whole number b, the symbolic expression $\frac{1}{b}$ is a *unit fraction* and represents the length of a quantity. When this quantity is iterated b times, the result is a length that is equal to the unit of measure. More generally, for a given unit of measure and whole numbers a and b, the expression $\frac{a}{b}$ represents a fraction whose magnitude equals the length $\frac{1}{b}$ iterated a times.

In this definitional statement, the term ‘length’ can be replaced by any other measurable attribute of a quantity such as area, volume, mass, or time. For fractions as abstract numbers, the statement is as follows:

For a given unit and whole number b, the symbolic expression $\frac{1}{b}$ is a *unit fraction* and represents a number. When it is iterated b times, the resulting number equals the unit. More generally, for a given unit and whole numbers a and b, the expression $\frac{a}{b}$ represents a fraction whose magnitude equals $\frac{1}{b}$ iterated a times.

In classroom-based studies, the need surfaces naturally for a lexical item to name the magnitude of a subunit of measure (POWELL, 2019). The subunit facilitates measuring and naming and eventually symbolizing the measure of the length when it is not an exact multiple of the unit of measure.

Finally, here is one way to initiate a measuring-perspective module on how a fraction gets its name. A start is to engage students in measuring the length of different objects available in a classroom. It may be necessary to convey the meaning of length as an attribute of objects and to clarify that measuring is a process to ascertain a count of how many iterations of a unit of measure equal an object’s length. The unit of measure can be the students’ choice. The class can discuss the inconsistent measures of an object’s length for varying units of measure and the efficacy of different units of measure given the extent of a particular length. Besides everyday objects, students can later be given

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Cuisenaire rods, so that they work in pairs to practice measuring the length of objects such as the sides of their desks. Students may benefit from discussing the relationship between the size of a unit of measure and the number of iterations it requires to measure a specific length. They may converge on the need to have a standard unit of measure and to have a name for the part that remains when the length of an object is not an exact multiple of the unit of measure.

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